On the local stability of semidefinite relaxations

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Joint work with **Sameer Agarwal** (Google),
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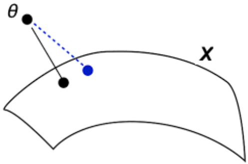
arXiv:1710.04287

Real Algebraic Geometry and Optimization - ICERM - 2018

Nearest point problems

Given a variety $X \subset \mathbb{R}^n$, and a point $\theta \in \mathbb{R}^n$,

$$\begin{aligned} \min_x \quad & \|x - \theta\|^2 \\ \text{s.t.} \quad & x \in X \end{aligned}$$



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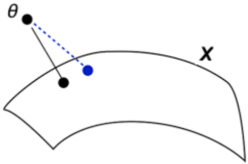

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- *SDP relaxations* have been successful in several applications.

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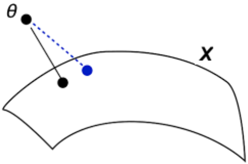

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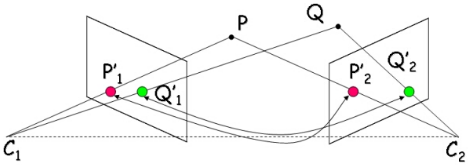
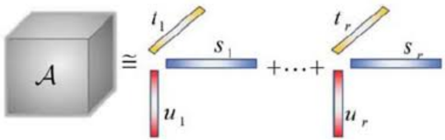
Goal

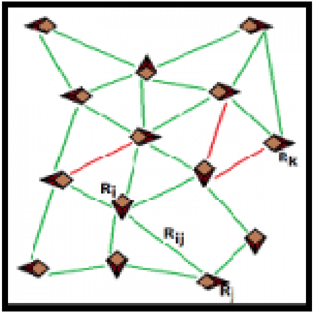
Study the behavior of SDP relaxations in the *low noise* regime: when x is sufficiently close to X.

Nearest point problems

Many different applications







Nearest point to the twisted cubic

$$\min_{x \in X} \quad \|x - \theta\|^2, \qquad \text{where} \qquad X := \{(x_1, x_2, x_3) : x_2 = x_1^2,\ x_3 = x_1 x_2\}$$

The twisted cubic X can be parametrized as $t \mapsto (t, t^2, t^3)$.

Its Lagrangian dual is the following SDP:

$$\max_{\gamma,\lambda_1,\lambda_2 \in \mathbb{R}} \quad \gamma, \qquad \text{s.t.} \qquad \begin{pmatrix} \gamma+\|\theta\|^2 & -\theta_1 & \lambda_1-\theta_2 & \lambda_2-\theta_3 \\ -\theta_1 & 1-2\lambda_1 & -\lambda_2 & 0 \\ \lambda_1-\theta_2 & -\lambda_2 & 1 & 0 \\ \lambda_2-\theta_3 & 0 & 0 & 1 \end{pmatrix} \succeq 0.$$

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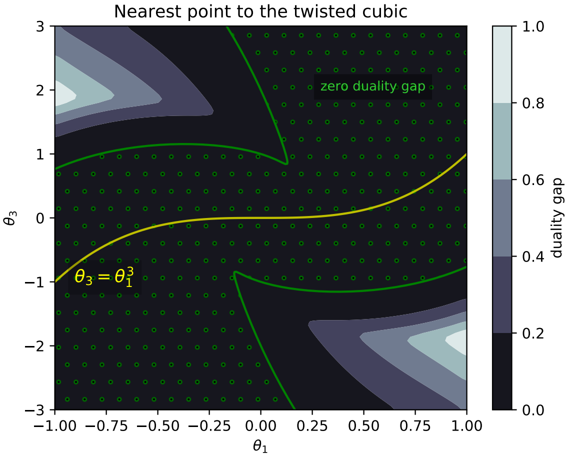
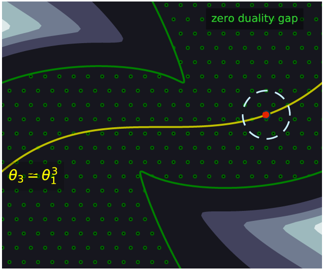


Nearest point problem to a quadratic variety

Theorem

If $\bar{\theta} \in X$ is a regular point then there is zero-duality-gap for any $\theta \in \mathbb{R}^n$ that is sufficiently close to $\bar{\theta}$.



Applications:

- Triangulation problem [Aholt-Agarwal-Thomas]
- Nearest (symmetric) rank one tensor

Parametrized QCQPs

Consider a family of *quadratically constrained programs* (QCQPs):

$$\begin{aligned} \min_{x\in\mathbb{R}^N} \quad & g_\theta(x) \\ & h^i_\theta(x) = 0 \quad \text{for } i = 1,\ldots,m \end{aligned} \tag{P_θ}$$

where g_θ, h^i_θ are *quadratic*, and the dependence on θ is *continuous*.
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Example: For a nearest point problem

$$g_\theta(x) := \|x - \theta\|^2, \qquad h^i(x) \text{ independent of } \theta$$

The problem is trivial for any $\bar{\theta} \in X$.

SDP relaxation of a (homogeneous) QCQP

Primal problem

$$\min_{x \in \mathbb{R}^N} \quad x^T G_\theta x \qquad\qquad (P_\theta)$$

$$x^T H_\theta^i x = b_i \quad i = 1, \ldots, m$$

Dual problem

$$\max_{\lambda \in \mathbb{R}^m} \quad d(\lambda) := -\textstyle\sum_i \lambda_i b_i \qquad\qquad (D_\theta)$$

$$\mathcal{Q}_\theta(\lambda) \succeq 0$$

where $\mathcal{Q}_\theta(\lambda)$ is the Hessian of the Lagrangian

$$\mathcal{Q}_\theta(\lambda) := G_\theta + \sum_i \lambda_i H_\theta^i \in \mathbb{S}^N.$$

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Problem statement

Assume that $\text{val}(P_{\bar\theta}) = \text{val}(D_{\bar\theta})$, i.e., $\bar\theta$ is a *zero-duality-gap* parameter.
Find conditions under which $\text{val}(P_\theta) = \text{val}(D_\theta)$ when θ is close to $\bar\theta$.

Characterization of zero-duality-gap

Given x_θ primal feasible, its *Lagrange multipliers* are:

$$\lambda \in \Lambda_\theta(x_\theta) \iff \lambda^T \nabla h_\theta(x_\theta) = -\nabla g_\theta(x_\theta) \iff \mathcal{Q}_\theta(\lambda) x_\theta = 0.$$

Lemma

Let $x_\theta \in \mathbb{R}^N, \lambda \in \mathbb{R}^m$. *Then* x_θ *is optimal to* (P_θ) *and* λ *is optimal to* (D_θ) *with* $\mathrm{val}(P_\theta) = \mathrm{val}(D_\theta)$ *iff:*

1. $h_\theta(x_\theta) = 0$ *(primal feasibility).*
2. $\mathcal{Q}_\theta(\lambda) \succeq 0$ *(dual feasibility).*
3. $\lambda \in \Lambda_\theta(x_\theta)$ *(complementarity).*

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Proof.

If $\mathcal{Q}_\theta(\lambda) x_\theta = 0$ and $h_\theta(x_\theta) = 0$, then

$$0 = x_\theta^T \mathcal{Q}_\theta(\lambda) x_\theta = x_\theta^T G_\theta x_\theta + \sum_i \lambda_i \, x_\theta^T H_i x_\theta = g_\theta(x_\theta) - d(\lambda).$$

Characterization of zero-duality-gap

Lemma

Let $\bar{\theta}$ be a zero-duality-gap parameter with $(\bar{x}, \bar{\lambda})$ primal/dual optimal. Assume that

1. $\mathcal{Q}_{\bar{\theta}}(\bar{\lambda})$ *has corank-one (strict-complementarity)*
2. $\exists x_\theta$ *feasible for* $(P_\theta), \lambda_\theta \in \Lambda_\theta(x_\theta)$ *s.t.* $(x_\theta, \lambda_\theta) \xrightarrow{\theta \to \bar{\theta}} (\bar{x}, \bar{\lambda})$.

Then there is zero-duality-gap when θ is close to $\bar{\theta}$.

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- $\mathcal{Q}_\theta(\lambda_\theta) \succeq 0$, so there is zero-duality-gap.

Nearest point to a quadratic variety

$$\min_{x \in X} \quad \|x - \theta\|^2, \qquad \text{where} \qquad X := \{x \in \mathbb{R}^n : f_1(x) = \cdots = f_m(x) = 0\}$$

Theorem

Let $\bar{\theta}$ be a regular point of X, i.e. $\operatorname{rank} \nabla f(\bar{\theta}) = \operatorname{codim} X$. Then there is zero-duality-gap for θ close to $\bar{\theta}$.

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- Need to find $\lambda_\theta \in \Lambda_\theta(x_\theta)$ s.t. $\lambda_\theta \xrightarrow{\theta \to \bar{\theta}} 0$.
- Regularity implies $\|\lambda_\theta\| \leq \frac{2}{\sigma(\nabla f)} \|x_\theta - \theta\| \xrightarrow{\theta \to \bar{\theta}} 0$. □

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□

Remark: The theorem generalizes to the case of *strictly convex* objective.

Guaranteed region of zero-duality-gap

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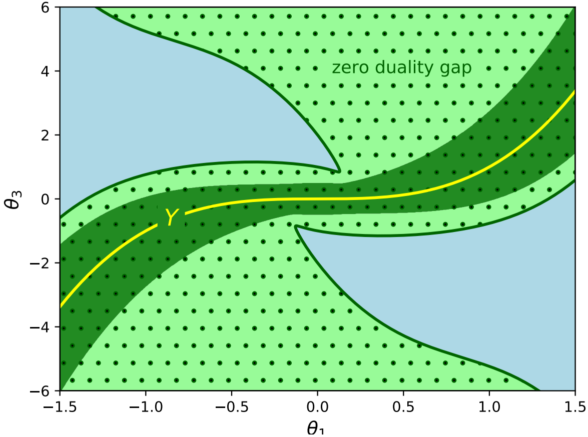

Application: Triangulation [Aholt-Agarwal-Thomas]

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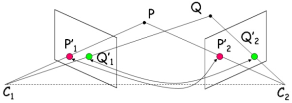
Given noisy images $\hat{u}_j \in \mathbb{R}^2$ of an unknown point,

$$\min_{u \in U} \quad \sum_j \|u_j - \hat{u}_j\|^2$$

where U is the *multiview variety* of the cameras.



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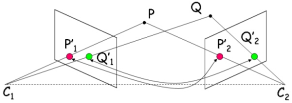

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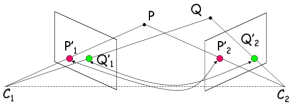

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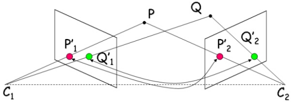

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- The regularity condition is easy to check.
- Under *low noise* the SDP relaxation is tight.

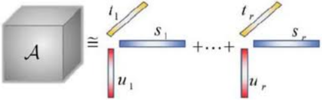
Application: Rank one approximation

Problem

Given a *tensor* $\hat{x} \in \mathbb{R}^{n_1 \times \cdots \times n_\ell}$, consider

$$\min_{x \in X} \quad \|x - \hat{x}\|^2$$

where X is the variety of rank one tensors (Segre).



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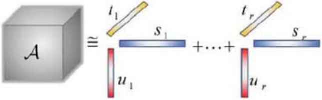

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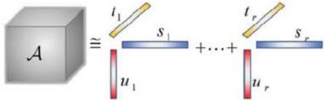


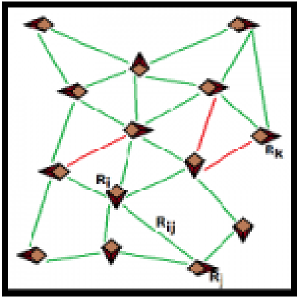
- The Segre variety is defined by quadratics (2×2 minors).
- Thus, the SDP relaxation is tight under low noise.

Application: Rotation synchronization

Problem

Given a graph $G = (V, E)$ and matrices $\hat{R}_{ij} \in \mathbb{R}^{d \times d}$ for $ij \in E$,

$$\min_{R_1,\dots,R_n \in SO(d)} \sum_{ij \in E} \|R_j - \hat{R}_{ij} R_i\|_F^2$$



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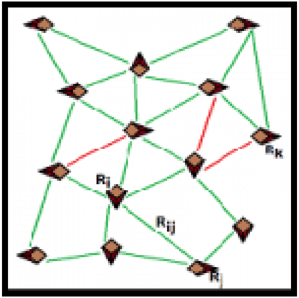

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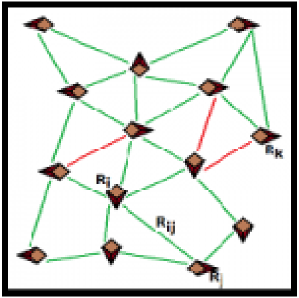

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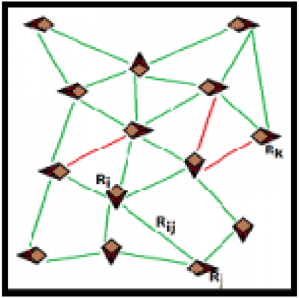

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- Similar tightness results have been shown [Fredriksson-Olsson], [Rosen-Carlone-Bandeira-Leonard], [Wang-Singer].

Application: Orthogonal Procrustes

Problem

Given matrices $A \in \mathbb{R}^{m_1 \times n}$, $B \in \mathbb{R}^{m_1 \times m_2}$, $C \in \mathbb{R}^{k \times m_2}$,

$$\min_{X \in St(n,k)} \|AXC - B\|_F^2$$

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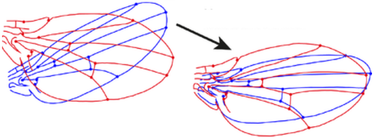

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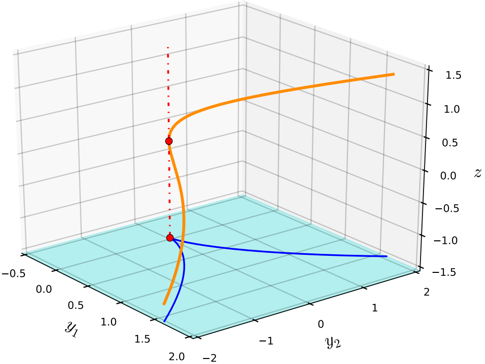
Nearest point to non-quadratic varieties

Any variety can be described by quadratics by using *auxiliary* variables.

Example: Nearest point problem to the curve $y_2^2 = y_1^3$ can be written as

$$\min_{y\in\mathbb{R}^2, z\in\mathbb{R}} \quad \|y-\theta\|^2, \quad \text{s.t.} \quad y_2 = y_1 z, \quad y_1 = z^2, \quad y_2 z = y_1^2.$$

The objective is *not strict convex*.



Stability of SDP relaxations of (arbitrary) QCQPs

Consider a general family of QCQPs:

$$\min_{x \in \mathbb{R}^N} \quad g_\theta(x) \qquad\qquad (P_\theta)$$
$$h_\theta^i(x) = 0 \quad \text{for } i = 1, \ldots, m$$

Let $\bar{\theta}$ be a zero-duality-gap parameter: $\text{val}(P_{\bar{\theta}}) = \text{val}(D_{\bar{\theta}})$.

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- There are bad cases, where the SDP relaxation is *non-informative*.
- We introduce a "Slater-type" condition that *guarantees* zero-duality-gap nearby $\bar{\theta}$.

Stability of SDP relaxations of QCQPs

Let $\bar{\theta}$ be a zero-duality-gap parameter with $(\bar{x}, \bar{\lambda})$ primal/dual optimal.

Assumption (restricted Slater)

There is $\mu \in \mathbb{R}^m$ s.t. the quadratic function $\Psi_\mu(x) := \sum_i \mu_i h^i_{\bar{\theta}}(x)$ satisfies: $\nabla \Psi_\mu(\bar{x}) = 0$, and Ψ_μ is strictly convex on $\ker \mathcal{Q}_{\bar{\theta}}(\bar{\lambda})$.

Theorem

Under the restricted Slater assumption and some regularity conditions, there is zero-duality-gap when θ is close to $\bar{\theta}$. Moreover, the SDP recovers the minimizer.

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Applications (ongoing):

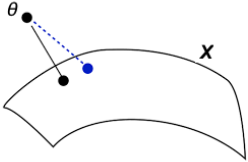
- Higher levels of SOS/Lasserre hierarchy.
- For instance: system identification, noisy deconvolution, camera resectioning, homography estimation, approximate GCD.

Using geometry to derive smaller SDP relaxations

Primal problem

$$\min_{x \in X} \quad x^T G x$$

$$X = \{x : x^T H^i x = b_i \text{ for } i = 1, \ldots, m\}$$



Dual problem

$$\max_{\lambda \in \mathbb{R}^m, Q \in \mathbb{S}^N} \quad -\textstyle\sum_i \lambda_i b_i$$

$$Q = G + \textstyle\sum_i \lambda_i H^i$$

$$Q \succeq 0$$

Using geometry to derive smaller SDP relaxations

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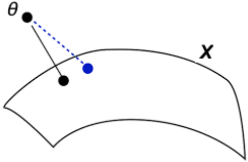

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Dual problem

Let $\hat{x}_1, \cdots, \hat{x}_S \in X$

$$\max_{\lambda \in \mathbb{R}^m, Q \in \mathbb{S}^N} \quad -\textstyle\sum_i \lambda_i b_i$$

$$\hat{x}_j^T Q \hat{x}_j = \hat{x}_j^T G \hat{x}_j + \textstyle\sum_i \lambda_i \hat{x}_j^T H^i \hat{x}_j \quad \text{for } j = 1, \ldots, S$$

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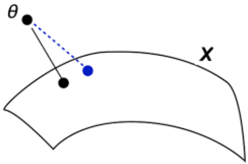

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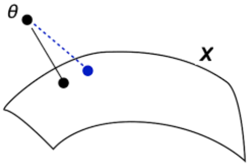

Dual problem

Let $\hat{x}_1, \cdots, \hat{x}_S \in X$

$$\max_{\gamma \in \mathbb{R}, Q \in \mathbb{S}^N} \quad -\gamma$$

$$\hat{x}_j^T Q \hat{x}_j = \hat{x}_j^T G \hat{x}_j + \gamma \quad \text{for } j = 1, \ldots, S$$

$$Q \succeq 0$$

- SDP is smaller, e.g., the multipliers $\lambda \in \mathbb{R}^m$ disappear.
- relaxation is stronger.

Example: Orthogonal Procrustes

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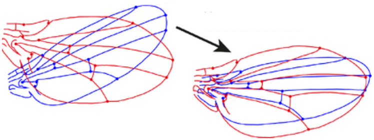

n	r	Equations SDP variables	Equations SDP constraints	Equations SDP time(s)	Gröbner basis (s)	Sampling SDP variables	Sampling SDP constraints	Sampling SDP time(s)
5	3	682	233	0.65	0.03	137	130	0.11
6	4	1970	576	1.18	9.94	326	315	0.14
7	5	4727	1207	3.56	-	667	651	0.24
8	6	9954	2255	13.88	-	1226	1204	0.45
9	7	19028	3873	42.14	-	2081	2052	1.10
10	8	33762	6238	124.43	-	3322	3285	2.48

Summary

- We analyzed the local stability of SDP relaxations.
- Found sufficient conditions for zero-duality-gap nearby $\bar{\theta}$.
- Many applications (triangulation, rank one approximation, rotation synchronization, orthogonal Procrustes).

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If you want to know more:

- D. Cifuentes, S. Agarwal, P. Parrilo, R. Thomas, *On the local stability of semidefinite relaxations*, `arXiv:1710.04287`.
- D. Cifuentes, C. Harris, B. Sturmfels, *The geometry of SDP-exactness in quadratic optimization*, `arXiv:1804.01796`.
- D. Cifuentes, P. Parrilo, *Sampling algebraic varieties for sum of squares programs*, `arXiv:1511.06751`.

Thanks for your attention!